Error propagation in a model of impact fracture

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Sensitivity to the perturbation in the initial condition of a model of impact fracture is observed. Numerical simulations show that the small perturbation in the initial condition is expanded following a power law by the dynamics of the model. The propagation of the effect of the perturbation inside a fractured object during the crack propagation is discussed. [S1063-651X(98)03810-0]

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The fracture process of brittle solid materials has been widely studied because of its importance in various fields of science and technology [1–3]. In spite of these studies, it is still very difficult to predict exactly how an object is fractured by an impact fracture process under given conditions. In view of classical physics, for a prediction of the behavior of a system, we need the dynamics governing the time evolution of the system and observation of the initial condition of the system with sufficient accuracy. In the case of the impact fracture process, our understanding of the dynamics of crack propagation is not very good for such predictions nor do we have a good technique to observe the fields of stress, strain, elastic moduli, etc. inside the object as the initial condition given to the crack propagation dynamics. Moreover, even with a full understanding of dynamics and a good observational technique, we would not be able to eliminate the difficulty of the prediction of the impact fracture process if the dynamics causes ''chaotic behavior.''

The chaotic dynamics has a characteristic nature to expand a minor perturbation in the initial condition exponentially [4]. Namely, perturbation ϵ_0 in the initial condition is expanded with time t as $\epsilon_0 \exp(\lambda t)$, where the constant λ is called the Lyapunov exponent. This means that the behavior of a chaotic system is very sensitive to its initial condition. That is, very slight observational error ϵ_0 in the initial condition smaller than the observational resolution is expanded far greater than the observational resolution in a very short time to be observed as a big difference of the system behavior. Since the observational error is inevitable in any observation and it is impossible to know the difference between the observed value and the real value smaller than the observational resolution, the long-term behavior of a chaotic system is unpredictable though the dynamics is deterministic.

In this way, whether the dynamics of crack propagation causes chaotic behavior is a big problem for prediction of impact fracture process. For quantitative analyses of the sensitivity to the small perturbation of the impact fracture by an experiment, we have to prepare two identical samples that have the same spatial distribution of compositions, textures, potential flaw, and so on, and we have to put a small perturbation of these fields onto one of the two samples. Then, identical impact must be loaded onto these two samples, and the crack propagation inside the two samples during the impact fracture process must be observed and compared to estimate the sensitivity to the perturbation. Obviously, it is impossible to conduct such an experiment by our present technique. Computer simulations resolve these technical problems.

In this paper, we observe the sensitivity to the perturbation of an impact fracture process using a specific model proposed by Inaoka and Takayasu [5].

The model was proposed to reproduce the fractality in the three-dimensional impact fracture such as the power-law fragment size distribution observed in an experiment of the fragmentation of sphere-shaped glass [6]. Not only does the model successfully reproduce the power-law fragment size distribution with good accuracy, it also explains universality of the distribution independent of the material of a fractured specimen [5,7] and the transition in the distribution by fragmentation of a plate-shaped specimen [8,9]. Though the derivation of the dynamics of the model from standard fracture mechanics is still incomplete, by these successes, we think that the model captures essential features of the impact fracture process and that the results obtained by this specific model are reliable.

The details of our numerical model are discussed in Ref. [5]. Here, we briefly review the basic idea. It has been recognized that a competition process among growing elements is generally essential for a growth system to produce fractality [10]. In a system of diffusion-limited aggregation [11], a typical fractal growth system, for example, the competition among growing clusters, produces a power-law cluster size distribution [12]. The analogy of the crack propagation to the diffusion-limited aggregation has been discussed for a quasistatic, elastic system [13,14], where the speed of the crack propagation is much smaller than that of the speed of the elastic wave in the medium. Our fracture model is based on a competitive process during crack propagation as follows.

The mechanisms of three-dimensional crack propagation in general situations are very complicated, so we focus on the situation where a rectangular parallelepiped object suffers an impact on one of its sides. It is a reasonable assumption that a planelike compressive density wave propagates from the hit surface toward the opposite side with a constant velocity v_0 as schematically shown in Fig. 1. We assume that most of the crack propagation occurs only on this density

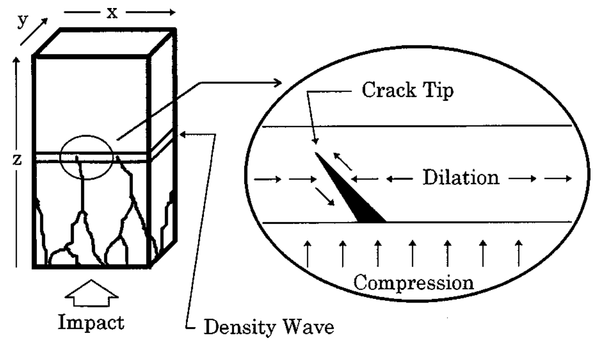

FIG. 1. A schematic figure of our numerical model. A thin compressive density wave causes dilation on the density wave plane, which drives the crack propagation.

wave, forming a so-called failure wave [15]. By this assumption we can treat the three-dimensional impact fracture process by considering a time evolution of a crack configuration on the two-dimensional density wave. We observe the propagation of cracks on a coordinate whose x-y plane is placed on the hit surface of the fractured object and the z axis is directed toward the propagating direction of the wave. The density wave is placed at a depth $v_0 t$ from the hit surface at time t, and in a small time interval Δt it proceeds a small distance $v_0 \Delta t$. In our model, time is discretized and a crack tip configuration on a layer at depth $v_0 t$ determines the configuration of the next layer at depth $v_0(t+\Delta t)$. By setting the constant time unit Δt such that $v_0 \Delta t$ is unity, the time t is measured by the depth z of the density wave from the hit surface. We call the time measured by the discretized depth z a time step. The density wave plane is also discretized, and the model is defined on a two-dimensional triangular lattice representing the density wave plane.

The dynamics of the crack tips is ruled by stress and strain field in the material. We introduce randomness of material by introducing spatial randomness in the elastic moduli. As schematically shown in Fig. 1, the randomness causes random dilation $\theta(x,y,z)$ on the density wave plane when the material undergoes compression by the density wave. This dilation is a driving force of the crack propagation. Starting from a standard description of deformation by a strain tensor field and neglecting the z component of the deformation, the two-dimensional deformation on the density wave plane is shown to be approximately described by a displacement potential $\phi(x,y,z)$ [5]. We assume that the boundary condition of $\phi(x,y,z)$ on crack tips is $\phi(x,y,z)=0$, which means that displacement near a crack tip is perpendicular to the crack tip. The boundary of the whole system is connected by periodic boundary conditions. Solving the equation for $\phi(x,y,z)$ and comparing the magnitude of displacement on both sides of the crack, the crack tip is assumed to move toward the place where the displacement is smaller. By this dynamics a larger fragment section tends to expand its area faster than neighboring small fragment sections, where a fragment section is defined by an area on the density wave surrounded by cracks. This kind of dynamics favorable for the larger fragment sections causes the competition between the fragment sections, which is the origin of the power-law fragment size distribution.

As one can see from above discussions, the dynamics of crack propagation in our model is not directly related to the potential flaw criterion, which is considered to be a basic concept of fracture mechanics [16–18]. Potential flaws may control the crack propagation dynamics with the effect of strain and stress fields. However, since we have not succeeded in deducing the dynamics rigorously from the standard fracture dynamics, we apply the macroscopic dynamics deduced using the knowledge of fractal growth discussed above in our numerical model.

The numerical simulation is started from the initial crack configuration such that the hit surface of the specimen is heavily destroyed and separated into many small fragments by many cracks. By the dynamics discussed above, the cracks can only move and merge on the propagating density wave with no bifurcation and no spontaneous stop. As a result, the crack density monotonically decreases as the density wave propagates. Such decay of crack density is quite natural in view of the energy because the crack propagation is a dissipative process.

For analyses of sensitivity of the system to small perturbation, we prepare two systems that have completely identical initial crack configurations but slightly different dilation fields $\theta(x,y,z)$. The dilation field of system 1, $\theta_1(x,y,z)$, is given by assigning a uniform random number in the range $[1.0,1.5)$ for each lattice site (x,y,z). The dilation field of system 2, $\theta_2(x,y,z)$, is given by that of system 1, $\theta_1(x,y,z)$, and a perturbation field $\epsilon(x,y,z)$ as

$$\theta_2(x,y,z)=\theta_1(x,y,z)+\epsilon(x,y,z). \tag{1}$$

We will check the behavior of the system for two different types of perturbation.

(a) The perturbation is given at one single lattice site on the impact surface $z=0$. That is, $\epsilon(x,y,z)$ is set zero except for $\epsilon(0,0,0)$, which is set 0.1.

(b) The perturbation is applied by a random field. For each site (x,y,z), $\epsilon(x,y,z)$ is assigned by a uniform random number in the range $[0.0,5.0\times 10^{-6})$.

Given an identical crack configuration on the hit surface $z=0$, the time evolution of the two systems is performed according to the process discussed above. Since the initial crack configurations on $z=0$ planes are identical for both systems 1 and 2, the crack pattern on the $z=0$ planes completely overlaps when the planes are overlapped. As time goes on the dynamics expands the effect of the perturbation given in the system 2, which causes the difference in the crack patterns on the density wave planes. This difference is observed as the unoverlapped patterns of cracks on the density wave planes. During the time evolution, we observe the expansion of the effect of perturbation by a Humming distance $d_H(z)$. The Humming distance $d_H(z)$ used in this paper is defined as the length of unoverlapped cracks on the density wave plane of system 1 normalized by total length of cracks on the density wave plane of the system 1. Note that the Humming distance $d_H(z)$ is dimensionless by definition. At the initial condition $d_H(0)=0$ and we expect that $d_H(z)$ increases with time.

We observed the behavior of the Humming distance $d_H(z)$ by simulations of the model of size 512×512 for 128 time steps. The numerical results in this paper are averages of results by five different simulations. The results of the time evolution of the Humming distance $d_H(z)$ are given in

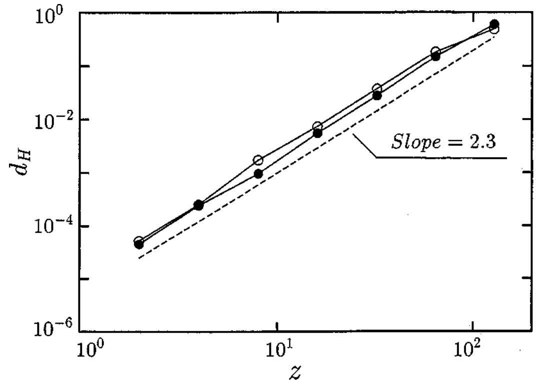

FIG. 2. The Humming distance d_H (dimensionless) by the single point perturbation (○) and the field perturbation (●) as functions of system depth z (lattice units).

Fig. 2 in log-log scale. As in the figure, the points are clearly on straight lines in the range of z from 10^0 to 10^2, indicating that the Humming distance $d_H(z)$ expands following a power law as

$$d_H(z) \propto z^{2.3} \tag{2}$$

for both types of perturbations (a) and (b). The power-law expansion of the Humming distance $d_H(z)$ is rather peculiar in view of the chaotic dynamics because most chaotic systems show an exponential expansion of the initial error. However, it can be interpreted as chaotic behavior with a zero Lyapunov exponent. The power-law expansion means that our model is at the marginal point between a chaotic and a nonchaotic system. Figure 3 shows a three-dimensional configuration of unoverlapped cracks for the case of the single point perturbation (a). This figure describes the manner of the propagation of the effect of perturbation from the perturbed point by the spatial pattern of unoverlapped cracks. Starting from the perturbed point on the hit surface, the unoverlapped cracks spread, forming a conelike shape.

The power-law expansion of the Humming distance $d_H(z)$ is determined by this conelike propagation of the effect of the perturbation. For the case of the single point perturbation (a), we observed the distance $r(z)$ between the projection point of the perturbed point on the density wave plane $(x,y,z)=(0,0,z)$ and the farthest unoverlapped crack on the density wave plane from the projection point as a

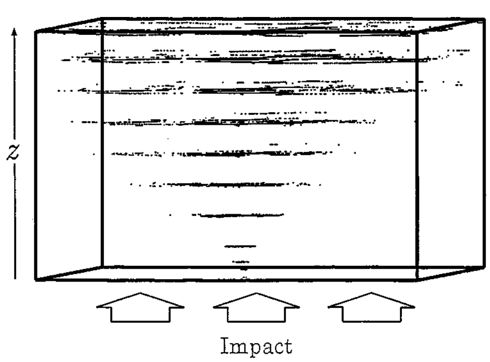

FIG. 3. A three-dimensional configuration of unoverlapped cracks produced by the single point perturbation.

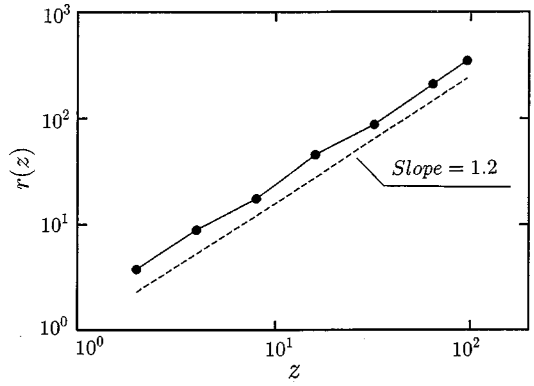

FIG. 4. The distance $r(z)$ (lattice units) as a function of z (lattice units) in log-log scale.

function of z. The $r(z)$ roughly describes the radius of the base of the cone formed by the unoverlapped cracks in Fig. 3 as a function of z. As in Fig. 4, $r(z)$ expands with a power law:

$$r(z) \propto z^{1.2}. \tag{3}$$

As mentioned above, the effect of the single point perturbation reaches within a circle of radius $r(z)$, and all of the unoverlapped cracks are inside the circle of radius $r(z)$ on the density wave plane. Assuming that the crack configuration on the density wave plane shows no complex fractality and its fractal dimension is 2 for a large observational scale, we introduce crack density, or crack length per unit area, $\sigma(z)$ on the density wave plane. Note that such crack density cannot be physically meaningful for complex fractal geometry in general. The crack density $\sigma(z)$ is a monotonically decreasing function of z. The total length of cracks l_t and the length of unoverlapped cracks l_u on the density wave plane in the system of size L at depth z are roughly written using $\sigma(z)$ as $l_t = L^2\sigma(z)$ and $l_u = \pi r(z)^2 \sigma(z)$, respectively. Thus, by definition, the Humming distance $d_H(z)$ defined in our paper has a simple meaning: $d_H = l_u/l_t = \pi r(z)^2/L^2$ represents the area where the effect of the perturbation appears normalized by the total area of density wave plane. In this case, we can expect a relation

$$d_H \propto r(z)^2 \propto z^{2.4}, \tag{4}$$

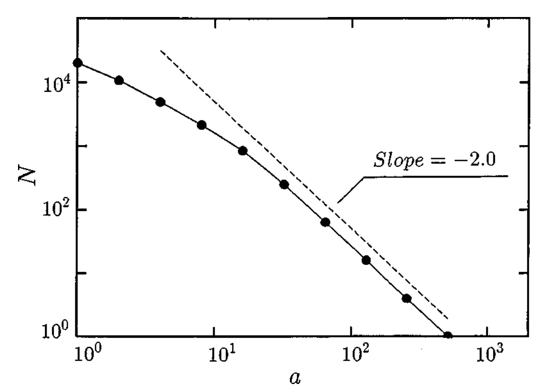

FIG. 5. The relation between the box size a (lattice units) and the number of the occupied box N by the box-counting method applied to the crack configuration on the density wave plane at $z = 64$. The fractal dimension is close to 2 for large scale.

which is close to Eq. (2). To check the assumption that the crack configuration on the density wave plane has no complex fractality, the result of fractal analysis of the crack configuration on the density wave plane by box-counting method is shown in Fig. 5. As we can see from the figure, the fractal dimension of the crack configuration on the density wave plane is close to 2 for a large observational scale. In this way, the power exponent of the Humming distance expansion is determined by the exponent of the propagation of the effect of perturbation and not by the complex fractality of the crack configuration.

The most significant result of our simulation is the fact that the Humming distance $d_H(z)$ does not expand exponentially as in most chaotic systems but expands with a power law. As mentioned above, it can be interpreted as chaotic behavior with a zero Lyapunov exponent. A negative Lyapunov exponent means that the perturbation in the initial condition vanishes during the evolution of the system in short time, while a positive exponent means that the initial perturbation expands exponentially and makes the behavior of the system unpredictable. In the case of the zero Lyapunov exponent, the initial perturbation does not vanish nor does it expand exponentially, but the effect of the perturbation remains in the system long time period and expands slowly. So, the impact fracture is more predictable than the usual chaotic behavior with a positive Lyapunov exponent, but it is still difficult to predict the exact behavior of the impact fracture because of the power-law expansion of errors. Some self-organized fractal system such as a model of river network formation by water erosion also shows power-law expansion of initial error [19]. Our results imply that dynamics of systems spontaneously organizing spatial fractality may show power-law expansion of initial error in general.

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